PLEASE ANSWER ALL QUESTIONS.
PLEASE EXPLAIN YOUR ANSWERS.

1. **Nash Equilibrium and Subgame-perfect Nash Equilibrium.**

(a) Denote the normal-form game below by G. Solve G by iterated elimination of strictly dominated strategies. Explain briefly each step (1 sentence).

		Player 2		
		t_1	t_2	t_3
Player 1	s_1	$2,4$	$3,5$	$1,3$
	s_2	$3,3$	$6,1$	$3,2$
	s_3	$4,2$	$2,1$	$4,0$
	s_4	$1,4$	$4,4$	$2,5$

Solution: s_4 is dominated by s_2. After eliminating s_4, then t_3 is dominated by t_1. After eliminating t_3, then s_1 is dominated by s_2. After eliminating s_1, then t_2 is dominated by t_1. After eliminating t_2, then s_2 is dominated by s_3. Solution: (s_3, t_1).

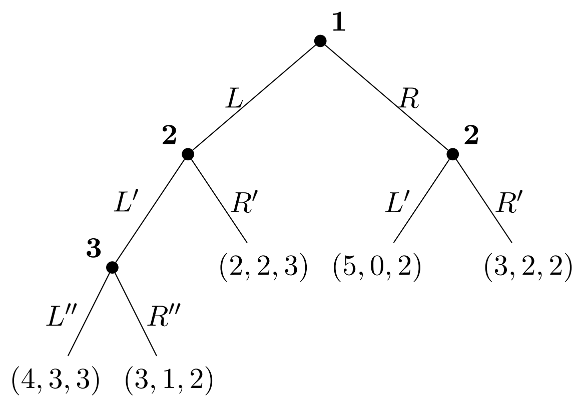
□

(b) Suppose we repeat G twice. Denote the resulting game by $G(2)$. Find the set of Subgame-perfect Nash Equilibria of $G(2)$. Be careful to write out the equilibrium strategies. (*Hint*: No new calculations are required.)

Solution: Since there is a unique outcome of the iterated elimination of strictly dominated strategies, this is the unique NE. Hence, it must be played in every subgame of the finitely repeated game. $SPNE = \{(\text{play } (s_3, t_1) \text{ in every subgame})\}$.

□

(c) Consider the extensive-form game given by the following game tree (the first payoff is that of player 1, the second payoff that of player 2, etc.):



i. Is this a game of perfect or imperfect information? How many proper subgames are there (excluding the game itself)? What are the strategy sets of the three players?

ii. Find all (pure strategy) Subgame-perfect Nash Equilibria.

iii. Is the strategy profile $(R, R'R', R'')$ a Nash Equilibrium? Discuss briefly (max. 3 sentences).

Solution: Perfect information. 3 proper subgames. $S_1 = \{L, R\}$. $S_2 = \{L'L', L'R', R'L', R'R'\}$. $S_3 = \{L'', R''\}$. Since perfect, complete information, we can solve by backward induction to get $SPNE = \{(L, L'R', L'')\}$. The strategy profile $(R, R'R', R'')$ is NE but rests on off-equilibrium-path 'threats' that are not credible.

□

(d) Consider again the game in (c), but suppose now that player 2 does not observe the move of player 1.

i. Draw the resulting game tree.

ii. Is this a game of perfect or imperfect information? How many proper subgames are there (excluding the game itself)? What are the strategy sets of the three players?

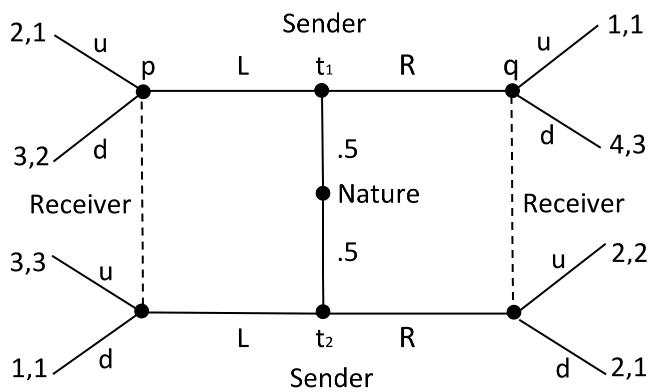
iii. Find all (pure strategy) Subgame-perfect Nash Equilibria. Discuss briefly (max. 3 sentences).

Solution: P2 now has a single information set. There is 1 proper subgame. Strategy sets as before, except $S_2 = \{L', R'\}$. P3's subgame gives $s_3 = L''$. Substituting this into the game we get:

		Player 2 L'	Player 2 R'
Player 1	L	$4, \underline{3}$	$2, 2$
Player 1	R	$\underline{5}, 0$	$\underline{3}, \underline{2}$

Thus: $SPNE = \{(R, R', L'')\}$. Now, P1 has an incentive to deviate to R if P2 plays L', making it impossible to have a SPNE where P1 plays L.

2. **Signaling.** Consider the following signaling game.



(a) Find all the (pure strategy) separating Perfect Bayesian Equilibria (PBE).

Solution: $(LR, du; p = 1, q = 0)$ and $(RL, ud; p = 0, q = 1)$ are PBE.

Case 1. Suppose $m(t_1) = L$ and $m(t_2) = R$. Then $p = 1$ and $q = 0$. This implies $a(L) = d$ and $a(R) = u$. Can check that $u_S(L, d; t_1) \geq u_S(R, u; t_1)$ and $u_S(R, u; t_2) \geq u_S(L, d; t_2)$ hold. Hence: PBE.

Case 2. Suppose $m(t_1) = R$ and $m(t_2) = L$. Then $p = 0$ and $q = 1$. This implies $a(L) = u$ and $a(R) = d$. Can check that $u_S(R, d; t_1) \geq u_S(L, u; t_1)$ and $u_S(L, u; t_2) \geq u_S(R, d; t_2)$ hold. Hence: PBE.

□

(b) Find the (pure strategy) pooling equilibrium in which both types send message L. Does it satisfy signaling requirement 5 (SR5)? Does it satisfy signaling requirement 6 (SR6)? Explain briefly (2-3 sentences).

Solution: Suppose $m(t_1) = m(t_2) = L$. Then $p = 1/2$ and $q \in [0, 1]$. Thus $a(L) = u$ since $(1/2)(1) + (1/2)(3) \geq (1/2)(2) + (1/2)(1)$. Notice $u_S(L, u; t_2) \geq u_S(R, a(R); t_2)$

always, but $u_S(L,u;t_1) \geq u_S(R,a(R);t_1)$ only if $a(R) = u$. In order for $a(R) = u$ to be optimal, we need

$$q(1) + (1-q)(2) \geq q(3) + (1-q)(1) \Leftrightarrow q \leq 1/3.$$

Hence: $(LL, uu; p = 1/2, q \leq 1/3)$ is PBE.

There is no strict dominance relationship so the PBE satisfies SR5. On the other hand, R is equilibrium dominated for t_2 but not for t_1. SR6: $q = 1$, implying that the PBE does not satisfy SR6.

□

(c) Suppose you are a used-cars salesman and you want to prove that your cars of high quality (quality is unobserved by customers, but known by you).

i. Give an example of a signal that is not credible and explain briefly (1 sentence) why it is not credible.

ii. Give an example of a signal that is credible and explain briefly (1 sentence) why it is credible.

Solution: Ad lib. □

3. **Nash bargaining.** Suppose two friends, Anne and Peter, have bought a piece of land of size 10 with the idea of building each of them a summerhouse on the land. They bargain over how much land each of them should get. Peter's utility from getting x_P units of land is:

$$u_P(x_P) = x_P.$$

Anne, on the other hand, has a larger family and therefore needs more space, so her utility from x_A units of land is

$$u_A(x_A) = 3x_A.$$

If they cannot reach an agreement, they don't get to build their summerhouse, so $x_A = x_P = 0$.

(a) Represent the situation as a bargaining problem, i.e. draw the sets X and U, and mark the disagreement points. Describe the Pareto efficient allocations.

Solution: $X = \{(x_A, x_P) | x_A, x_P \geq 0, x_A + x_P \leq 10\}$ and $U = \{(v_A, v_P) | v_A, v_P \geq 0, v_A + 3v_P \leq 30\}$, with $d = (0,0)$. The efficient set of allocations $\{(x_A, x_P) | x_A, x_P \geq 0, x_A + x_P = 10\}$ with efficient payoffs $\{(v_A, v_P) | v_A, v_P \geq 0, v_A + 3v_P = 30\}$.

□

(b) Determine the Nash bargaining solution of the game. What are the allocations of land to Anne and Peter?

Solution: Solution must be efficient: $v_A = 30 - 3v_P$. Substitute this into the Nash objective function to get $v_P(30 - 3v_P)$. Take FOC to get $30 - 6v_P = 0 \Rightarrow v_P^* = 5$. This gives $v_A^* = 30 - 3 \cdot 5 = 15$. Since $(v_A^*, v_P^*) \in U$ and $(v_A^*, v_P^*) \geq d$, this is the solution. The allocations are $x_A^* = x_P^* = 5$.

□

(c) Assume now that the dictatorial president of the summerhouse community decides that in case of dispute, the summerhouse community will take 40% of the land and use it for building communal parks. The remaining 60% is for Anne and Peter, but the president has decided that Anne should get twice as much as Peter, because of her large family. What is the new disagreement point? What is the Nash bargaining solution of the new game? What are the allocations? Briefly explain the difference to (b) (max. 3 sentences).

Solution: Now $d = (4 \cdot 3, 2) = (12, 2)$. Substituting the efficiency constraint into the objective function we get $(v_P - 2)(30 - 3v_P - 12)$. Take the FOC to get $24 - 6v_P = 0 \Rightarrow v_P^* = 4$. This implies $v_A^* = 30 - 3 \cdot 4 = 18$. The allocations are $x_A^* = 6$ and $x_P^* = 4$. Improving the disagreement point for Anne improves her bargaining position, and therefore her bargaining outcome.